Faculty of Science, Technology, Engineering and Mathematics
M373 Optimization

M373 Optimization

Diagnostic Quiz

Contents

Are you ready for M373?

This diagnostic quiz is designed to help you decide if you are ready to study *Optimization* (M373). This document also contains some advice on preparatory work that you may find useful before starting M373 (see below and page 6). The better prepared you are for M373 the more time you will have to enjoy the mathematics, and the greater your chance of success.

The topics which are included in this quiz are those that we expect you to be familiar with before you start the module. If you have previously studied *Pure mathematics* (M208), *Mathematical methods and modelling* (MST210) or *Mathematical methods* (MST224) then you should be familiar with most of the topics covered in the quiz.

We suggest that you try this quiz first without looking at any books, and only look at a book when you are stuck. You will not necessarily remember everything; for instance, for the calculus questions you may need to look up a set of rules for differentiation or use the table of standard derivatives provided on page 2. This is perfectly all right, as such tables are provided in the Handbook for M373. You need to check that you are able to use them though.

If you can complete the quiz with only the occasional need to look at other material, then you should be well prepared for M373. If you find some topics that you remember having met before but need to do some more work on, then you should consider the suggestions for refreshing your knowledge on page 6.

Try the questions now, and then see the notes on page 6 of this document to see if you are ready for M373.

Do contact your Student Support Team via StudentHome if you have any queries about M373, or your readiness to study it.

Table of standard derivatives

Function	Derivative
a	0
x^a	ax^{a-1}
e^{ax}	ae^{ax}
$\ln(ax)$	$\frac{1}{x}$
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(ax)$	$a\sec^2(ax)$
$\cot(ax)$	$-a\mathrm{cosec}^2(ax)$
$\sec(ax)$	$a\sec(ax)\tan(ax)$
$\mathrm{cosec}(ax)$	$-a\mathrm{cosec}(ax)\cot(ax)$
$\arcsin(ax)$	$\frac{a}{\sqrt{1-a^2x^2}}$
$\arccos(ax)$	$-\frac{a}{\sqrt{1-a^2x^2}}$
$\arctan(ax)$	$\frac{a}{1+a^2x^2}$
$\mathrm{arccot}(ax)$	$-\frac{a}{1+a^2x^2}$
$\mathrm{arcsec}(ax)$	$\frac{a}{\lvert ax\rvert\sqrt{a^2x^2-1}}$
$\mathrm{arccosec}(ax)$	$-\frac{a}{\lvert ax\rvert\sqrt{a^2x^2-1}}$

Diagnostic quiz questions

1 Vectors

Question 1

If $\mathbf{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ find

(a) $\mathbf{a} + \mathbf{b}$, (b) $3\mathbf{a} - 2\mathbf{b}$, (c) $|\mathbf{a}|$, (d) $\mathbf{a} \cdot \mathbf{b}$.

Question 2

If $\mathbf{x} = [3, 1, -2, 4]^T$ and $\mathbf{y} = [-2, 5, 1, 3]^T$ find

(a) $2\mathbf{x} + 3\mathbf{y}$, (b) $|\mathbf{x}|$, (c) $\mathbf{x}^T\mathbf{y}$.

Question 3

Find the vector equation of the line segment between the points with position vectors $\mathbf{p} = [2, -3, 1]^T$ and $\mathbf{q} = [5, -2, -3]^T$.

2 Matrices

Question 4

Let $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 5 & 4 \\ 2 & 0 \\ 1 & 3 \end{bmatrix}$.

Calculate each of the following, or state why the calculation is not possible.

(a) $2\mathbf{A} + \mathbf{B}$ (b) $\mathbf{A} - 3\mathbf{C}$ (c) $\mathbf{AB}$ (d) $\mathbf{AC}$ (e) $\mathbf{CA}$ (f) $\mathbf{BC}^T$

Question 5

Calculate the determinants of the following matrices.

(a) $\begin{bmatrix} 2 & 4 \\ -3 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 1 & 0 \\ 2 & -3 & 2 \\ 4 & 2 & -1 \end{bmatrix}$

Question 6

Calculate the inverse of the matrix $\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$.

Question 7

Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 1 & -1 \\ -4 & -2 \end{bmatrix}$.

3 Simultaneous equations

Question 8

Solve, if possible the following systems of equations.

(a) $2x + 3y + z = 5$
$x - 4y + 2z = 11$
$3x + y - 3z = 2$

(b) $x - 5y + 2z = 4$
$2x + y + 3z = -2$
$4x - 9y + 7z = 0$

(c) $x + 3y - 2z = 1$
$2x - y + 3z = 2$
$4x + 5y - z = 4$

4 Differentiation

Question 9

Differentiate the following functions with respect to x.

(a) $f(x) = 3x^4 + 2\cos x - 4\ln x$

(b) $g(x) = \sin 3x + e^{2x}$

(c) $h(x) = x^5 \tan 2x$

(d) $p(x) = \dfrac{2x^3 + 3x + 1}{2x + 1}$

(e) $q(x) = \cos(2x^3 + 1)$

Question 10

Find the second derivative with respect to t of $f(t) = 2t^5 + 3\ln(2t)$.

Question 11

Find and classify the stationary points of $y = x^3 - 3x^2 - 9x + 6$.

5 Partial differentiation

Question 12

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ when $f(x, y) = 3x^2 \sin y + \log(x^2y^4)$.

Question 13

Find all the second partial derivatives of $g(x, y) = 2x^4 \cos(x + 2y)$.

6 Taylor series

Question 14

Calculate the Taylor series about $\pi/2$ for the function $f(x) = \sin 2x$, giving sufficient terms to make the general pattern clear.

7 Vector calculus

Question 15

If $f(x, y, z) = x^2z + yz^2 + xy^2$, find $\boldsymbol{\nabla} f$ (also known as grad$f$) at the point $(1, 3, -2)$.

Hence find the rate of change of f in the direction $[1, 2, -2]^T$ at the point $(1, 3, -2)$.

8 Sequences

Question 16

What is the third term of the sequence given by

$$x_1 = 1, x_i = x_{i-1}^2 - 4, \qquad (i = 2, 3, 4, \ldots) \ ?$$

What can I do to prepare for M373?

Now that you have finished the quiz, how did you get on? The information below should help you to decide what to do next.

Most of this quiz was unfamiliar to me. I don't think I have met any of the topics before. →	You should consider taking M208, MST210 or MST224 before M373. Contact your Student Support Team to discuss this.
Some of these topics are familiar to me, but not all of them. →	If you have met less than 75% of these topics then you should consider taking M208, MST210 or MST224 before M373. Contact your Student Support Team to discuss this.
I have met all these topics before, but I am a bit rusty on some, and had to look up how to do a few of them. →	You will benefit from doing some revision before starting M373. See below for possible sources of help.
I can do all these questions. →	You are probably well prepared for M373, but do brush up on any topics you need to further enhance your confidence. M373 uses the Maxima computer algebra system. You may wish to prepare for the module by studying the M373 "Introduction to Maxima" which is available to registered students in the M373 area of `learn1.open.ac.uk/site/maxima`. You might also wish to get ahead with your M373 studies. Some early units of M373 are available to registered students under DISCOVER, then **Discover your module** at `learn2.open.ac.uk/site/s-maths`.

If you have any queries contact your Student Support Team via StudentHome.

What resources are there to help me prepare for M373?

If you need to do some background preparation before starting M373, we suggest you concentrate your efforts on the following topics.

- Linear algebra (matrices and vectors)
- Differentiation (both for functions of one variable and functions of several variables)
- Vector calculus (particularly the gradient of a function of several variables).

There are many undergraduate textbooks covering these topics. The following are available to registered students online through the OU Library.

- Lipschutz, S. and Lipson, M., “Schaum’s Outline of Linear Algebra”, Fourth Edition, McGraw-Hill, ISBN: 9780071543521.
- Hefferon, J., “Linear Algebra”, ISBN: 9781616100537.
- Bear, H.S., “Understanding calculus”, IEEE-Wiley.
- Ayres, F. and Mendelson, E., “Schaum’s Outline of Calculus”, Fifth edition, McGraw-Hill, ISBN: 9780071508612
- Wrede,R. and Spiegel, M., “Schaum’s Outline of Advanced Calculus”, Third edition, McGraw-Hill, ISBN: 9780071623667.

If you have studied M208, MST210, or MST224, then you could use parts of these to revise for M373.

The mathcentre web-site (`www.mathcentre.ac.uk`) includes several teach-yourself books, summary sheets, revision booklets, online exercises and video tutorials on a range of mathematical skills.

Diagnostic quiz solutions

Solution to question 1

(a) $\mathbf{a}+\mathbf{b} = \begin{bmatrix}3\\4\end{bmatrix} + \begin{bmatrix}2\\-5\end{bmatrix} = \begin{bmatrix}5\\-1\end{bmatrix}.$

(b) $3\mathbf{a}-2\mathbf{b} = 3\begin{bmatrix}3\\4\end{bmatrix} - 2\begin{bmatrix}2\\-5\end{bmatrix} = \begin{bmatrix}9\\12\end{bmatrix} - \begin{bmatrix}4\\-10\end{bmatrix} = \begin{bmatrix}5\\22\end{bmatrix}.$

(c) $|\mathbf{a}| = \sqrt{3^2+4^2} = \sqrt{25} = 5.$

(d) $\mathbf{a}\cdot\mathbf{b} = \begin{bmatrix}3\\4\end{bmatrix} \cdot \begin{bmatrix}2\\-5\end{bmatrix} = 3\times 2 + 4\times(-5) = 6-20 = -14.$

Solution to question 2

(a)
$$\begin{aligned} 2\mathbf{x}+3\mathbf{y} &= 2[3,1,-2,4]^T + 3[-2,5,1,3]^T \\ &= [6,2,-4,8]^T + [-6,15,3,9]^T \\ &= [0,17,-1,17]^T. \end{aligned}$$

(b) $|\mathbf{x}| = \sqrt{3^2+1^2+(-2)^2+4^2} = \sqrt{30}.$

(c)
$$\begin{aligned} \mathbf{x}^T\mathbf{y} &= [3,1,-2,4]\begin{bmatrix}-2\\5\\1\\3\end{bmatrix} \\ &= 3\times(-2) + 1\times 5 + (-2)\times 1 + 4\times 3 \\ &= 9. \end{aligned}$$

Note that $\mathbf{x}^T\mathbf{y} = \mathbf{x}\cdot\mathbf{y}$.

Solution to question 3

The vector equation of a line is of the form $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ where $\mathbf{d}$ is a vector in the direction of a line.

In particular, the vector equation a a line between the points with position vectors $\mathbf{p}$ and $\mathbf{q}$ is $\mathbf{x} = \mathbf{p} + t(\mathbf{q}-\mathbf{p})$, where the $\mathbf{p}$ is the point corresponding to $t=0$ and $\mathbf{q}$ the point corresponding to $t=1$.

Here, $\mathbf{q}-\mathbf{p} = [5,-2,-3]^T - [2,-3,1]^T = [3,1,-4]^T$.

So the required line segment is

$$\mathbf{x} = [2,-3,1]^T + t[3,1,-4]^T, \qquad 0 \le t \le 1$$

or, equivalently

$$\mathbf{x} = [2+3t, -3+t, 1-4t]^T, \qquad 0 \le t \le 1.$$

Solution to question 4

(a) $2\mathbf{A}+\mathbf{B} = 2\begin{bmatrix}2 & 3\\-1 & 4\end{bmatrix} + \begin{bmatrix}3 & -1\\2 & 5\end{bmatrix} = \begin{bmatrix}4 & 6\\-2 & 8\end{bmatrix} + \begin{bmatrix}3 & -1\\2 & 5\end{bmatrix} = \begin{bmatrix}7 & 5\\0 & 13\end{bmatrix}.$

(b) $\mathbf{A}-3\mathbf{C}$

This cannot be calculated since $\mathbf{A}$ has size 2×2 and $\mathbf{C}$ (hence $3\mathbf{C}$) has size 3×2, which differ.

(c) $\mathbf{AB} = \begin{bmatrix}2 & 3\\-1 & 4\end{bmatrix}\begin{bmatrix}3 & -1\\2 & 5\end{bmatrix} = \begin{bmatrix}2\times 3+3\times 2 & 2\times(-1)+3\times 5\\(-1)\times 3+4\times 2 & (-1)\times(-1)+4\times 5\end{bmatrix} = \begin{bmatrix}12 & 13\\5 & 21\end{bmatrix}.$

(d) $\mathbf{A}$ has size 2×2 and $\mathbf{C}$ has size 3×2. Since the number of columns of $\mathbf{A}$ is different to the number of rows of $\mathbf{C}$, $\mathbf{AC}$ cannot be calculated.

(e) $\mathbf{CA} = \begin{bmatrix} 5 & 4 \\ 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 31 \\ 4 & 6 \\ -1 & 15 \end{bmatrix}.$

(f) $\mathbf{BC}^T = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 5 & 2 & 1 \\ 4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 6 & 0 \\ 30 & 4 & 17 \end{bmatrix}.$

Solution to question 5

(a) $\det\left(\begin{bmatrix} 2 & 4 \\ -3 & 5 \end{bmatrix}\right) = 2 \times 5 - 4 \times (-3) = 22.$

(b)
$$\begin{aligned} \det\left(\begin{bmatrix} 3 & 1 & 0 \\ 2 & -3 & 2 \\ 4 & 2 & -1 \end{bmatrix}\right) &= 3\det\left(\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}\right) - 1\det\left(\begin{bmatrix} 2 & 2 \\ 4 & -1 \end{bmatrix}\right) + 0\det\left(\begin{bmatrix} 2 & -3 \\ 4 & 2 \end{bmatrix}\right) \\ &= 3\left((-3) \times (-1) - 2 \times 2\right) - 1 \times (2 \times (-1) - 2 \times 4) \\ &= 3 \times (-1) - (-10) \\ &= 7. \end{aligned}$$

Solution to question 6

Note, $\det\left(\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}\right) = 2$. So, $\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 2 & 1 \end{bmatrix}.$

Solution to question 7

The characteristic equation is given by $\begin{vmatrix} 1-\lambda & -1 \\ -4 & -2-\lambda \end{vmatrix} = 0,$

that is, $(1-\lambda)(-2-\lambda) - 4 = 0 \quad$ or, $\lambda^2 + \lambda - 6 = 0.$

This can be factorised as $(\lambda + 3)(\lambda - 2) = 0$, hence the eigenvalues are 2 and -3.

The eigenvector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ associated with the eigenvalue λ are found by solving

$$\begin{bmatrix} 1-\lambda & -1 \\ -4 & -2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

For $\lambda = 2$ this gives

$$\begin{bmatrix} -1 & -1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

which can be written as

$$\begin{aligned} -x - y &= 0 \\ -4x - 4y &= 0. \end{aligned}$$

These are satisfied when $x = -y$ hence an eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

For $\lambda = -3$ the equation gives

$$\begin{bmatrix} 4 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

which can be written as

$$\begin{aligned} 4x - y &= 0 \\ -4x + y &= 0. \end{aligned}$$

These are satisfied when $x = y/4$ hence an eigenvector is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Solution to question 8

(a) The system is

$$\begin{aligned} 2x+3y+\ z &= 5 && (1) \\ x-4y+2z &= 11 && (2) \\ 3x+\ y-3z &= 2. && (3) \end{aligned}$$

Eliminate z from equations (2) and (3).

$$\begin{aligned} & 2x+\ 3y+z = 5 && (4) \\ (2) - 2\times(1): \quad & -3x-10y \quad = 1 && (5) \\ (3) + 3\times(1): \quad & 9x+10y \quad = 17 && (6) \end{aligned}$$

Eliminate y from equations (5) and (6).

$$\begin{aligned} & 2x+\ 3y+z = 5 && (7) \\ & -3x-10y \quad = 1 && (8) \\ (5) + (6): \quad & 6x \qquad\quad = 18 && (9) \end{aligned}$$

So, from (9), $x = 3$, then using (8), $y = -1$ and from (7), $z = 2$.

(b) The system is

$$\begin{aligned} x-5y+2z &= 4 && (10) \\ 2x+\ y+3z &= -2 && (11) \\ 4x-9y+7z &= 0. && (12) \end{aligned}$$

Eliminate x from equations (11) and (12).

$$\begin{aligned} & x-\ 5y+2z = 4 && (13) \\ (11) - 2\times(10): \quad & 11y-\ z = -10 && (14) \\ (12) - 4\times(10): \quad & 11y-\ z = -16 && (15) \end{aligned}$$

Equations (14) and (15) are inconsistent, so the system has no solution.

(c) The system is

$$\begin{aligned} x+3y-2z &= 1 \qquad (16)\\ 2x-\ \ y+3z &= 2 \qquad (17)\\ 4x+5y-\ \ z &= 4. \qquad (18)\end{aligned}$$

Eliminate x from equations (17) and (18).

$$\begin{aligned} & x+3y-2z = 1 && (19)\\ (17) - 2\times(16): \quad & -7y+7z = 0 && (20)\\ (18) - 4\times(16): \quad & -7y+7z = 0 && (21)\end{aligned}$$

Equations (20) and (21) are identical, so there is one degree of freedom. Let $z = c$, a constant, then to satisfy (20) and (21), $y = c$. From equation (19), $x = 1 - c$.

So the family of solutions satisfying the system is $x = 1 - c, y = c, z = c$.

Solution to question 9

(a) $f'(x) = 12x^3 - 2\sin x - \dfrac{4}{x}$

(b) $g'(x) = 3\cos 3x + 2e^{2x}$

(c) $h'(x) = 5x^4 \tan 2x + 2x^5 \sec^2 2x$

(d) $$\begin{aligned} p(x) &= \frac{(2x+1)(6x^2+3) - (2x^3+3x+1)(2)}{(2x+1)^2}\\ &= \frac{12x^3+6x+6x^2+3-4x^3-6x-2}{(2x+1)^2}\\ &= \frac{8x^3+6x^3+1}{(2x+1)^2}\end{aligned}$$

(e) $q(x) = -6x^2\sin(2x^3+1)$

Solution to question 10

$f'(t) = 10t^4 + \dfrac{3}{t} = 10t^4 + 3t^{-1}, \quad$ so $f''(t) = 40t^3 - 3t^{-2}$.

Solution to question 11

Stationary points occur when $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

Here, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x - 9$, so we need to solve $3x^2 - 6x - 9 = 0$, or $x^2 - 2x - 3 = 0$.

Factorising gives $(x-3)(x+1) = 0$, so the stationary points occur at $x = -1$ and $x = 3$.

The nature of the points can be classified using the second derivative test.

Here, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x - 6$.

So, at $x = -1$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -12 < 0$ hence point is a local maximum.

At $x = 3$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12 > 0$ hence point is a local minimum.

Solution to question 12

$\frac{\partial f}{\partial x} = 6x \sin y + \frac{2}{x}$.

$\frac{\partial f}{\partial y} = 3x^2 \cos y + \frac{4}{y}$.

Solution to question 13

$g(x, y) = 2x^4 \cos(x + 2y)$.

The first partial derivatives are

$$\frac{\partial g}{\partial x} = 8x^3 \cos(x + 2y) - 2x^4 \sin(x + 2y)$$
$$\frac{\partial g}{\partial y} = -4x^4 \sin(x + 2y).$$

The second partial derivatives are

$$\frac{\partial^2 g}{\partial x^2} = 24x^2 \cos(x + 2y) - 16x^3 \sin(x + 2y) - 2x^4 \cos(x + 2y)$$
$$\frac{\partial^2 g}{\partial x \partial y} = -16x^3 \sin(x + 2y) - 4x^4 \cos(x + 2y)$$
$$\frac{\partial^2 g}{\partial y^2} = -8x^4 \cos(x + 2y)$$
$$\frac{\partial^2 g}{\partial y \partial x} = -16x^3 \sin(x + 2y) - 4x^4 \cos(x + 2y).$$

Solution to question 14

The Taylor series about a for the function $f(x)$ is

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots.$$

For $f(x) = \sin 2x$ we have $f(\pi/2) = 0$ and,

$$\begin{aligned} f'(x) &= 2\cos 2x & f'(\pi/2) &= 1 \\ f''(x) &= -4\sin 2x & f''(\pi/2) &= 0 \\ f'''(x) &= -8\cos 2x & f'''(\pi/2) &= -8 \\ f^{(iv)}(x) &= 16\sin 2x & f^{(iv)}(\pi/2) &= 0 \\ f^{(v)}(x) &= 32\cos 2x & f^{(iv)}(\pi/2) &= 32. \end{aligned}$$

So, $f(x) = 0 + 1\left(x - \frac{\pi}{2}\right) + 0\left(x - \frac{\pi}{2}\right)^2 - \frac{8}{3!}\left(x - \frac{\pi}{2}\right)^3 + 0\left(x - \frac{\pi}{2}\right)^4 + \frac{32}{5!}\left(x - \frac{\pi}{2}\right)^5$
$= \left(x - \frac{\pi}{2}\right) - \frac{8}{3!}\left(x - \frac{\pi}{2}\right)^3 + \frac{32}{5!}\left(x - \frac{\pi}{2}\right)^5$

The coefficient of $\left(x - \frac{\pi}{2}\right)^n$ is $\begin{cases} 0 & \text{if } n \text{ even} \\ \frac{2^n}{n!} & \text{if } n \text{ odd.} \end{cases}$

Solution to question 15

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]^T$$
$$= \left[2xz + y^2, z^2 + 2xy, x^2 + 2yz\right]^T$$

So at $(1, 3, -2)$, $\quad \nabla f(1, 3, -2) = [5, 10, -11]^T$

The rate of change of f in the direction $\hat{\mathbf{s}}$, where $\hat{\mathbf{s}}$ is a unit vector, is given by $\nabla f \cdot \hat{\mathbf{s}}$.

Here, the direction $\mathbf{s} = [1, 2, -2]^T$ hence $\hat{\mathbf{s}} = \frac{1}{3}[1, 2, -2]^T = \left[\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right]^T$.

So the rate of change of f in the direction of $\mathbf{s}$ at $(1, 3, -2)$ is

$$\nabla f \cdot \hat{\mathbf{s}} = [5, 10, -11]^T \cdot \left[\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right]^T = \frac{47}{3}.$$

Solution to question 16

For the sequence

$$x_1 = 1, x_i = x_{i-1}^2 - 4, \qquad (i = 1, 2, 3, \ldots)$$

we have:

$$x_1 = 1$$
$$x_2 = 1^2 - 4 = -3$$
$$x_3 = (-3)^2 - 4 = 5.$$

So the third term, x_3 is 5.