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Applications of probability (M343) Diagnostic Quiz

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1. Introduction

In embarking on Applications of probability (M343), basic mathematical competence is at least as important as basic knowledge of probability. This quiz covers a number of key mathematical skills with which you should definitely be familiar.

Any further special mathematical techniques you need are covered in M343. However, it would be helpful if you were familiar with differential equations and matrices, as covered in Stage 2 modules in mathematics.

For most of M343, you are also expected to have some knowledge of probability: probability functions and probability density functions; the binomial, Poisson, geometric, exponential and normal distributions; the Poisson process. This material is all covered in Analysing data (M248), though a thorough revision of it is included in the first book of M343. We recommend that you study this as early as you can so that you are prepared for the rest of the module.

(continued on following page)

Try each question for yourself, using a pencil and paper and your calculator where appropriate. Then click on the green section letter (e.g. '(a)') to see the solution. Click on the □ symbol at the end of the solution to return to the question. Use the ↑ and ↓ keys to move from Section to Section.

There is some advice on evaluating your performance at the end of the quiz.

2. Rounding and accuracy

Exercise 1.

Round 1.24743

(a) to two decimal places, and

(b) to four significant figures.

Exercise 2. In part (c) below, 8 is an exact whole number. The other numbers in the calculations have already been rounded. Use your calculator to work out the value of each expression and consider how many decimal places it is reasonable to give in your answer.

(a) 0.163×0.246

(b) 0.163×7.296

(c) $2.05 - 0.1237 \times 8$

3. Fractions

EXERCISE 3. Express each of the following over a suitable common denominator.

(a) $\frac{5}{6} - \frac{3}{8}$

(b) $\frac{-2}{2x+5} + \frac{1}{x-3}$

(c) $1 - \frac{4}{x^2}$

4. Formulas

Exercise 4.

(a) Evaluate the binomial coefficient $\binom{7}{3}$.

(b) Given the values $x_0 = 0$ and $y = 0.6$, calculate the values of x_1 and x_2 to four significant figures using the formula

$$x_{j+1} = \frac{4x_j + 7y}{4\,(1 - y)},$$

for $j = 0,\ 1$.

5. Powers and logarithms

Exercise 5. Write each of the following in the form x^n, where n is some number.

(a) $\dfrac{1}{x^2}$

(b) $x^2 \times x^3$

(c) $\left(x^2\right)^3$

(d) $\dfrac{x^2}{x^3}$

Exercise 6.

(a) Express the cube root of 0.3 as a power.

(b) Use your calculator to evaluate it to four decimal places.

Exercise 7. In M343, logarithms are always natural logs (to base e). This may be **ln** on your calculator.

Use the fact that $\log 3 = 1.0986$ and $\log 4 = 1.3863$ to four decimal places, to evaluate the following.

(a) $\log 12$

(b) $\log (3/4)$

(c) $\log 2$

Exercise 8. Simplify

(a) $\exp (2 \log x)$

(b) $\exp (-3 \log (1 - x))$

6. Matrices

Exercise 9. $\mathbf{A}$ is the 3×3 matrix $\begin{pmatrix} 0.5 & 0.1 & 0.4 \\ 0.6 & 0.0 & 0.4 \\ 0.7 & 0.3 & 0.0 \end{pmatrix}$, $\mathbf{v}$ is the column vector $\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{w}$ is the row vector $\begin{pmatrix} 4 & 1 & 3 \end{pmatrix}$.

Work out each of the following.

(a) $\mathbf{Av}$

(b) $\mathbf{wA}$

(c) $\mathbf{A}^2$

7. Differentiation

Exercise 10. Differentiate each of the following with respect to x.

(a) x^4 and $\dfrac{1}{x^2}$

(b) e^{-3x} and $\log\left(1+x^2\right)$

(c) xe^{-2x}

(d) $\dfrac{0.2x}{1-0.8x}$

8. Integration

EXERCISE 11. Work out each of the following integrals.

(a) $\displaystyle\int x^2 dx$ and $\displaystyle\int \frac{1}{x^3} dx$

(b) $\displaystyle\int_1^3 \left(1 + \frac{3}{x^2}\right) dx$

(c) $\displaystyle\int_0^2 \frac{2x}{x^2 + 4} dx$

(d) $\displaystyle\int_0^\infty x e^{-x} dx$

9. Differential equations

Exercise 12. Use the technique of separation of variables to find the general solutions of each of the following differential equations.

(a) $\dfrac{dy}{dx} = 2xy$

(b) $\dfrac{dy}{dx} = y + 5$

10. Post-mortem

You should be familiar with the techniques covered by Exercises 1 to 12 before embarking on M343.

If you found these exercises particularly difficult, and have not studied MST124 Essential Mathematics 1 (or its predecessor MST121 Using Mathematics), you would be well advised to consider taking this module before starting M343.

If you had difficulty only with the later parts of Exercises 10 and 11 and with Exercise 12, then this should not delay your registering for M343. However, you would be well advised to reinforce your understanding with the help of the relevant units in MST124, or any basic textbook on calculus.

All the differentiation techniques required for M343 are covered in Books B and C of MST124. Basic integration and the technique of separation of variables are covered in Book C of MST124. Integration by parts and integration by substitution are also covered in Book C of MST124.

If you have any queries about your suitability for the course, you should contact your Student Support Team via Student Home.

Solutions to Exercises

Exercise 1(a) 1.24743 rounded to two decimal places is 1.25.

Since the next digit after the '4' is 7, which is 5 or more, round up; the original number is closer to 1.25 than to 1.24.

□

Exercise 1(b) 1.24743 rounded to four significant figures is 1.247.

□

Exercise 2(a) 0.0401. A calculator gives 0.040098 but, since the original numbers are rounded to three significant figures, you cannot expect more than three significant figures in your answer to be reliable.

□

Exercise 2(b) 1.19. (A calculator gives 1.189248.) □

Exercise 2(c) 1.06. (A calculator gives 1.0604.) You cannot expect anything after the second decimal place to be reliable, because 2.05 is rounded to two decimal places.

□

Exercise 3(a) Here, the lowest common denominator is 24; (6 and 8 are both factors of 24).

$$\begin{aligned}\frac{5}{6}-\frac{3}{8} &= \frac{5\times 4}{24}-\frac{3\times 3}{24}\\ &= \frac{20-9}{24}\\ &= \frac{11}{24}.\end{aligned}$$

□

Exercise 3(b) The common denominator is $(2x+5)(x-3)$.

$$\begin{aligned} \frac{-2}{2x+5} + \frac{1}{x-3} &= \frac{-2\,(x-3)}{(2x+5)\,(x-3)} + \frac{2x+5}{(2x+5)\,(x-3)} \\ &= \frac{-2x+6+2x+5}{(2x+5)\,(x-3)} \\ &= \frac{11}{(2x+5)\,(x-3)}. \end{aligned}$$

□

Exercise 3(c) $1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2}$.

□

Exercise 4(a) $\binom{7}{3} = \frac{7\times 6\times 5}{1\times 2\times 3} = 35.$

□

Exercise 4(b) $x_1 = \frac{4.2}{1.6} = 2.625$ exactly; $x_2 = \frac{14.7}{1.6} = 9.188$, to four significant figures.

□

Exercise 5(a) Using the rule $\frac{1}{x^a} = x^{-a}$ gives

$$\frac{1}{x^2} = x^{-2}.$$

□

Exercise 5(b) Using the rule $x^a \times x^b = x^{a+b}$ gives

$$x^2 \times x^3 = x^5.$$

□

Exercise 5(c) Using the rule $(x^a)^b = x^{ab}$ gives

$$\left(x^2\right)^3 = x^6.$$

□

Exercise 5(d) Using the rule $\frac{x^a}{x^b} = x^{a-b}$ gives

$$\frac{x^2}{x^3} = x^{-1}.$$

□

Exercise 6(a) The cube root of 0.3 is $0.3^{\frac{1}{3}}$.

□

Exercise 6(b) $0.3^{\frac{1}{3}} = 0.6694$, to four decimal places.

You can either raise 0.3 to the power (1/3) on your calculator using the x^y (or y^x) button, or you can use the $\sqrt[x]{y}$ button, (if your calculator has one).

□

Exercise 7(a) Using the rule $\log(a \times b) = \log a + \log b$,

$$\begin{aligned} \log 12 &= \log(3 \times 4) \\ &= \log 3 + \log 4 \\ &= 1.0986 + 1.3863 \\ &= 2.485. \end{aligned}$$

(Note that we have expressed the solution to three decimal places. Had we expressed it to four decimal places, there may have been an error of ± 1 in the fourth decimal place due to rounding errors.)

□

Exercise 7(b) Using the rule $\log{(a/b)} = \log a - \log b$,

$$\begin{aligned} \log 3/4 &= \log 3 - \log 4 \\ &= 1.0986 - 1.3863 \\ &= -0.288. \end{aligned}$$

Recall that the logarithm of a number between 0 and 1 is negative (and $\log 1 = 0$).

(Note that we have expressed the solution to three decimal places. Had we expressed it to four decimal places, there may have been an error of ± 1 in the fourth decimal place due to rounding errors.)

□

Exercise 7(c) Using the rule $\log a^b = b \log a$,

$$\begin{aligned}\log 2 &= \log \sqrt{4} \\ &= \log 4^{1/2} \\ &= \frac{1}{2} \log 4 \\ &= \frac{1}{2} \times 1.3863 \\ &= 0.693.\end{aligned}$$

(Note that we have expressed the solution to three decimal places. Had we expressed it to four decimal places, there may have been an error of ± 1 in the fourth decimal place due to rounding errors.)

□

Exercise 8(a) Since $2\log x = \log x^2$,

$$\begin{aligned}\exp(2\log x) &= \exp\left(\log x^2\right)\\ &= x^2.\end{aligned}$$

This uses the fact that the logarithm function and the exponential function are inverse functions. So $\exp(\log x) = x$, for any $x > 0$. (Similarly, $\log(\exp x) = x$ for all x.)

□

Exercise 8(b)

$$\begin{aligned}\exp\left(-3\log\left(1-x\right)\right) &= \exp\left(\log\left(1-x\right)^{-3}\right)\\ &= \left(1-x\right)^{-3}\\ &= \frac{1}{\left(1-x\right)^{3}}.\end{aligned}$$

□

Exercise 9(a)

$$\mathbf{Av} = \begin{pmatrix} 3.3 \\ 3.6 \\ 3.1 \end{pmatrix}$$

The first entry is $(0.5 \times 4) + (0.1 \times 1) + (0.4 \times 3)$, using the first row of $\mathbf{A}$. The other two entries are obtained using the second and third rows of $\mathbf{A}$.

□

Exercise 9(b)

$$\mathbf{wA} = \begin{pmatrix} 4.7 & 1.3 & 2.0 \end{pmatrix}$$

The second entry, for example, is $(4 \times 0.1) + (1 \times 0.0) + (3 \times 0.3)$, using $\mathbf{w}$ and the second column of $\mathbf{A}$.

Observe that the answers to parts (a) and (b) are not the same. They do not even have the same shape.

□

Exercise 9(c)

$$\mathbf{A}^2 = \begin{pmatrix} 0.59 & 0.17 & 0.24 \\ 0.58 & 0.18 & 0.24 \\ 0.53 & 0.07 & 0.40 \end{pmatrix}$$

Matrices and vectors are used only in Unit 6 on Markov chains.

□

Exercise 10(a) The derivative of x^n is nx^{n-1}, so $\dfrac{d}{dx}\left(x^4\right) = 4x^3$.

Since $\dfrac{1}{x^2} = x^{-2}$, $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\right) = \dfrac{d}{dx}\left(x^{-2}\right) = -2x^{-3}$, or $\dfrac{-2}{x^3}$.

□

Exercise 10(b) The *composite function rule*, or *chain rule*, for differentiating the composition of one function with a second one is needed here. (For example, $\exp(-3x) = \exp(y)$, where $y = -3x$.)

The derivatives are $-3e^{-3x}$ and $\dfrac{2x}{1+x^2}$.

□

Exercise 10(c) This is the *product* of two functions. The derivative is

$$\begin{aligned}\frac{d}{dx}(x)\,e^{-2x}+x\frac{d}{dx}\left(e^{-2x}\right) &= 1e^{-2x}=x\,(-2)\,e^{-2x}\\ &= (1-2x)\,e^{-2x}.\end{aligned}$$

□

Exercise 10(d) This is the *quotient* of two functions. The derivative is

$$\begin{aligned}
& \frac{(1-0.8x)\,\frac{d}{dx}\,(0.2x) - 0.2x\,\frac{d}{dx}\,(1-0.8x)}{(1-0.8x)^2} \\
= \quad & \frac{(1-0.8x)\,0.2 - 0.2x\,(-0.8)}{(1-0.8x)^2} \\
= \quad & \frac{0.2 - 0.16x + 0.16x}{(1-0.8x)^2} \\
= \quad & \frac{0.2}{(1-0.8x)^2}.
\end{aligned}$$

□

Exercise 11(a) For $n \neq -1$, $\int x^n dx = \frac{x^{n+1}}{n+1} + c$, where c is an arbitrary constant of integration. So

$$\int x^2 dx = \frac{x^3}{3} + c.$$

Since $\frac{1}{x^3} = x^{-3}$,

$$\begin{aligned} \int \frac{1}{x^3} dx &= \int x^{-3} dx \\ &= \frac{x^{-2}}{-2} + c \\ &= -\frac{1}{2x^2} + c. \end{aligned}$$

□

Exercise 11(b)

$$\begin{aligned}\int_1^3 \left(1+\frac{3}{x^2}\right) dx &= \left[x-\frac{3}{x}\right]_1^3 \\ &= (3-1)-(1-3) \\ &= 4.\end{aligned}$$

□

Exercise 11(c) In general, $\displaystyle\int \frac{f'(x)}{f(x)} = \log f(x) + c.$

Observe that the numerator of the integrand, $(2x)$, is the derivative of the denominator, $\left(x^2+4\right)$. So

$$\begin{aligned}
\int_0^2 \frac{2x}{x^2+4}dx &= \left[\log\left(x^2+4\right)\right]_0^2 \\
&= \log 8 - \log 4 \\
&= \log\left(\frac{8}{4}\right) \\
&= \log 2 \\
&= 0.6931,
\end{aligned}$$

to four decimal places.

□

Exercise 11(d) Here, *integration by parts* is required. First, integrate e^{-x} to get $-e^{-x}$. Then

$$\begin{aligned}\int_0^\infty xe^{-x}dx &= \left[x\left(-e^{-x}\right)\right]_0^\infty - \int_0^\infty \frac{d}{dx}(x)\left(-e^{-x}\right)dx \\ &= 0 + \int_0^\infty e^{-x}dx \\ &= \left[-e^{-x}\right]_0^\infty \\ &= -(0-1) \\ &= 1.\end{aligned}$$

□

Exercise 12(a) Separating the variables, the equation is written in the form

$$\frac{dy}{y} = 2xdx.$$

Observe that x does not appear on the left-hand-side and y is not on the right-hand-side. Now integrate both sides to obtain

$$\log y = x^2 + C$$

or, taking exponentials,

$$y = K \exp x^2,$$

where C is an arbitrary constant and $K = e^C$.

Other techniques for solving differential equations are taught in the course where they are needed (mainly in Units 7 and 8).

□

Exercise 12(b) Separating the variables leads to

$$\frac{dy}{y+5} = dx.$$

Integrating both sides gives

$$\log(y+5) = x + C,$$

or, taking exponentials,

$$y = Ke^x - 5.$$

As before, C is an arbitrary constant and $K = e^C$.

□