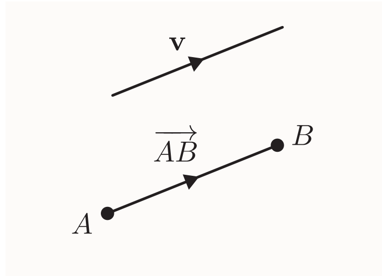
Vectors

A **vector** is a quantity that has both a size (usually called **magnitude**) and a direction. The following quantities are vectors:

- **Displacement**, the position of one point relative to another.
- **Velocity**, the speed of an object together with its direction.

A **displacement vector** is a vector that represents displacement.



Scalars

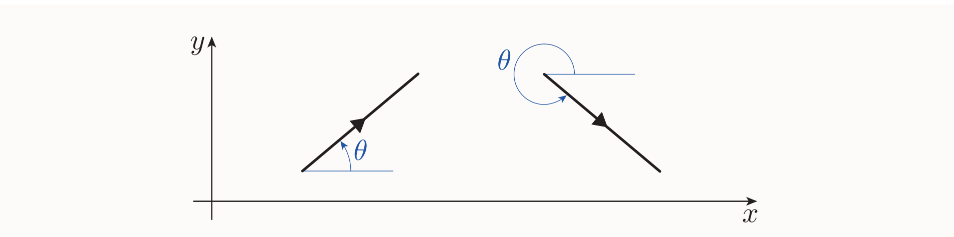
A **scalar** is a quantity that has size but no direction. The following quantities are scalars:

- Distance, the magnitude of displacement.
- Speed, the magnitude of velocity.
- Time, temperature and volume.

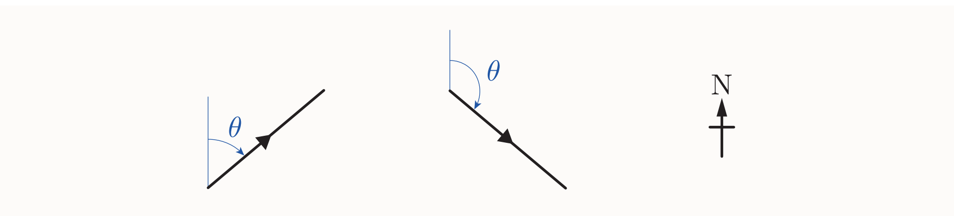
Direction of a vector

The **direction** of a vector is usually specified in one of the following two ways.

- As an angle measured anticlockwise from the positive direction of the x-axis to the direction of the vector.



- As a **bearing**; that is, the angle in degrees between 0° and 360° measured clockwise from north to the direction of the vector.



Heading and course of a ship or aircraft

For a moving ship or aircraft:

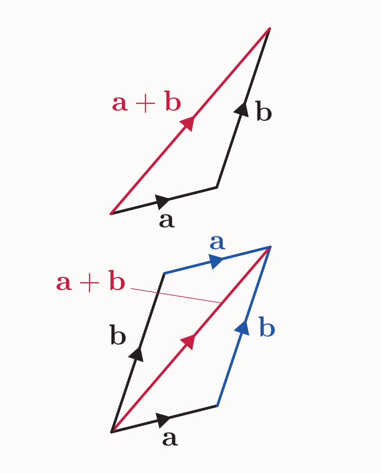
- Its **heading** is the direction in which it is pointing, given as a bearing.
- Its **course** is the direction in which it is actually moving.

These directions may be different, due to the effect of a current or wind.

The actual velocity of a ship or aircraft is the resultant of the velocity that it would have if the water or air were still, and the velocity of the current or wind.

Vector algebra

Two vectors are **equal** if they have the same magnitude and the same direction.



The zero vector

The **zero vector**, denoted by **0** (bold zero), is the vector whose magnitude is zero. It has no direction.

Triangle law for vector addition

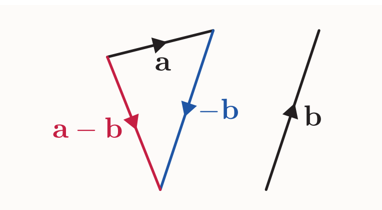
To find the **sum** (**resultant**) of two vectors **a** and **b**, place the tail of **b** at the tip of **a**. Then **a** + **b** is the vector from the tail of **a** to the tip of **b**.

Parallelogram law for vector addition

To find the sum of two vectors **a** and **b**, place their tails together, and complete the resulting figure to form a parallelogram. Then **a** + **b** is the vector formed by the diagonal of the parallelogram, starting from the point where the tails of **a** and **b** meet.

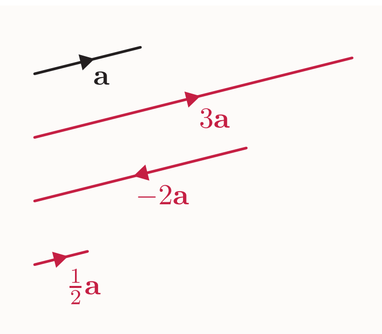
Negative of a vector

The **negative** of a vector **a**, denoted by −**a**, is the vector with the same magnitude as **a**, but the opposite direction.



Vector subtraction

To subtract **b** from **a**, add −**b** to **a**. That is, **a** − **b** = **a** + (−**b**).



Scalar multiple of a vector

Suppose that **a** is a vector. Then, for any non-zero real number m, the **scalar multiple** $m\mathbf{a}$ of **a** is the vector

- whose magnitude is $|m|$ times the magnitude of **a**
- that has the same direction as **a** if m is positive, and the opposite direction if m is negative.

Also, $0\mathbf{a} = \mathbf{0}$. (That is, zero times the vector **a** is the zero vector.)

Properties of vector algebra

These properties hold for all vectors **a**, **b** and **c**, and all scalars m and n.

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}$
6. $(m + n)\mathbf{a} = m\mathbf{a} + n\mathbf{a}$
7. $m(n\mathbf{a}) = (mn)\mathbf{a}$
8. $1\mathbf{a} = \mathbf{a}$

Representing vectors using component form

A **unit vector** is a vector with magnitude 1.

The **Cartesian unit vectors**, denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, are the vectors of magnitude 1 in the directions of the x-, y- and z-axes, respectively.

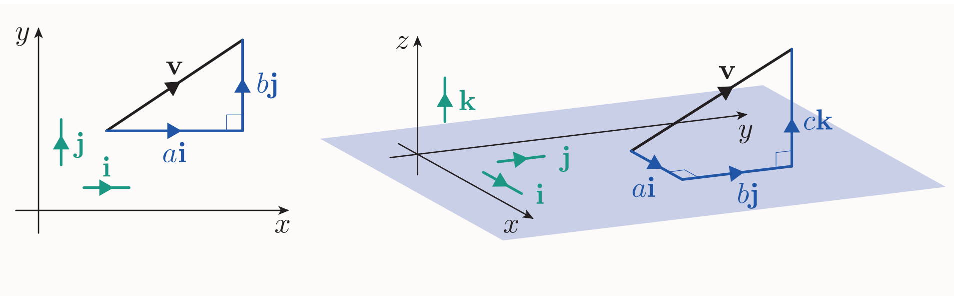
Component form of a vector

The **component form** of a two-dimensional vector $\mathbf{v}$ is the expression $a\mathbf{i} + b\mathbf{j}$, where $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$.

It can also be written as $\begin{pmatrix} a \\ b \end{pmatrix}$.

The **component form** of a three-dimensional vector $\mathbf{v}$ is the expression $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, where $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

It can also be written as $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.



The **i-component** and **j-component** (or the x-**component** and y-**component**) of a vector $\mathbf{v}$ are the scalars a and b, respectively, in the component form $a\mathbf{i} + b\mathbf{j}$ of $\mathbf{v}$. The components of a three-dimensional vector are referred to in a similar way.

A **column vector** is a vector written as a column, such as $\begin{pmatrix} a \\ b \end{pmatrix}$.

Vector algebra using component form

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$, and m is a scalar, then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} \qquad -\mathbf{a} = -a_1\mathbf{i} - a_2\mathbf{j}$$
$$\mathbf{a} - \mathbf{b} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j} \qquad m\mathbf{a} = ma_1\mathbf{i} + ma_2\mathbf{j}.$$

In column notation, if $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, and m is a scalar, then

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix} \qquad -\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix}$$
$$\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix} \qquad m\mathbf{a} = \begin{pmatrix} ma_1 \\ ma_2 \end{pmatrix}.$$

The algebra of three-dimensional vectors is similar.

Converting vectors from component form to magnitude and direction, and vice versa

To find the magnitude of a two- or three-dimensional vector from its components

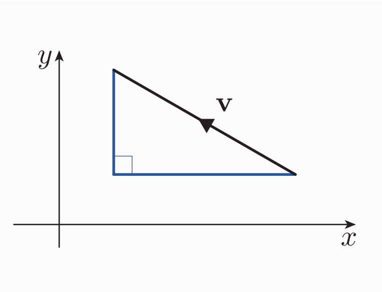
The two-dimensional vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ has magnitude

$$|\mathbf{v}| = \sqrt{a^2 + b^2}.$$

The three-dimensional vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ has magnitude

$$|\mathbf{v}| = \sqrt{a^2 + b^2 + c^2}.$$

To find the magnitude and direction of a two-dimensional vector (not parallel to an axis) from its component form



1. Using the components, sketch a right-angled triangle whose hypotenuse is the vector, and whose shorter sides are parallel to the x- and y-axes.
2. Use Pythagoras' theorem (or, equivalently, the formula above) to find the magnitude of the vector.
3. Use trigonometry to find an acute angle in the triangle.
4. Use this acute angle to work out the direction of the vector.

To find the component form of a two-dimensional vector (not parallel to an axis) from its magnitude and direction

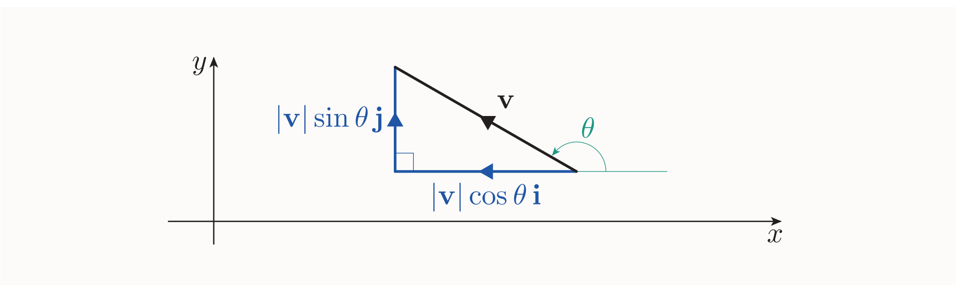
1. Using the magnitude and direction, sketch a right-angled triangle whose hypotenuse is the vector, and whose shorter sides are parallel to the x- and y-axes.
2. Use trigonometry to find the magnitudes of the components.
3. Use the direction of the vector to find the signs of the components.

An alternative method is to use the formula below.

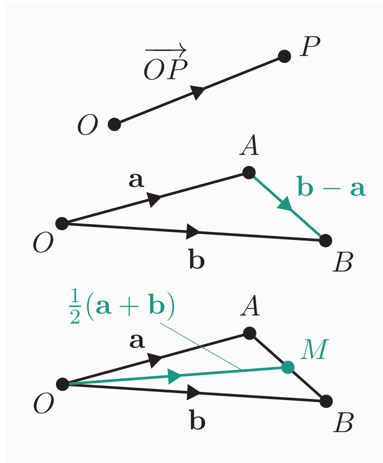
Component form of a two-dimensional vector in terms of its magnitude and the angle that it makes with the positive x-direction

If the two-dimensional vector $\mathbf{v}$ makes the angle θ with the positive x-direction, then

$$\mathbf{v} = |\mathbf{v}|\cos\theta\,\mathbf{i} + |\mathbf{v}|\sin\theta\,\mathbf{j}.$$



Position vectors



The **position vector** of a point P is the displacement vector $\overrightarrow{OP}$, where O is the origin.

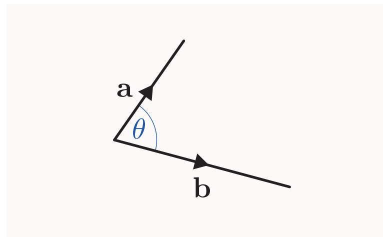
If the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, then

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}.$$

Midpoint formula in terms of position vectors

If the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, then the midpoint of the line segment AB has position vector $\frac{1}{2}(\mathbf{a}+\mathbf{b})$.

Scalar product



The angle between two non-zero vectors is the angle θ in the range $0 \leq \theta \leq 180°$ between their directions when the vectors are placed tail to tail.

Scalar product of two vectors

The **scalar product** (or **dot product**) of the non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta,$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

If $\mathbf{a}$ or $\mathbf{b}$ is the zero vector, then $\mathbf{a}\cdot\mathbf{b} = 0$.

Properties of the scalar product

These properties hold for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, and every scalar m.

1. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are non-zero. If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, then $\mathbf{a}\cdot\mathbf{b} = 0$, and vice versa.
2. $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$
3. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
4. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$
5. $(m\mathbf{a})\cdot\mathbf{b} = m(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(m\mathbf{b})$

Scalar product in terms of components

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$, then

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2.$$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

To find the angle between two vectors in component form

The angle θ between two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|},$$

where $0 \leq \theta \leq 180°$.